PHYSICAL REVIEW E **77**, 055305(R) (2008)

Efficiency of cargo towing by a microswimmer

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(Received 26 January 2008; published 29 May 2008)

We study the properties of an arbitrary microswimmer towing a passive load through a viscous liquid. The simple close-form expression for the dragging efficiency of a microswimmer dragging a distant load is found, and the approximation for finite mutual proximity is derived. We show that, while the swimmer can be arbitrarily efficient, the dragging efficiency is always bounded from above. It is also demonstrated, that opposite to Purcell's assumption [E. M. Purcell, Proc. Natl. Acad. Sci. U.S.A. **94**, 11307 (1997)], the hydrodynamic coupling can "help" the propeller to tow the load. We support our conclusions by rigorous numerical calculations for the rotary swimmer, towing a spherical cargo positioned at a finite distance.

DOI: 10.1103/PhysRevE.77.055305 PACS number(s): 47.63.mf

In recent years there has been an increasing interest in propulsion at low Reynolds numbers, both theoretically [1–10] and experimentally [11]. These and other works have improved our understanding of the basic properties of locomotion on small scales. However, it is not sufficient to understand the mechanisms and properties of free microswimmers alone—it is necessary to estimate the performance of these swimmers as *propellers* that tow a useful cargo, e.g., a therapeutic load or miniature camera. This question, which attracted only limited attention, had already been shown to have some interesting answers: Purcell [1] had studied the particular case of a rotating helix pushing a spherical particle under the assumption of negligible hydrodynamic interactions. He showed that due to the structure of a grand-resistance matrix, which connects the force and torque on a body to its translational and angular velocities, the optimal rotating propeller should have the same size as the load.

In this paper we address arbitrary shaped swimmers and loads, and investigate the effect of their mutual hydrodynamic interaction on the performance of the swimmer as a load propeller. We find that, while propellers that can enclose a load within may theoretically have arbitrarily high efficiency (consider, for instance, the "treadmiller" [8]), the dragging efficiency of a swimmer towing a remote load is always bounded from above, and there is an optimal propeller-load size ratio, which depends on the propeller efficiency and their mutual proximity. We also show, that in contrast to Purcell's assumption [1], there are cases when hydrodynamic coupling between the load and the propeller enhances the dragging efficiency, and also provide a criterion for the optimal cargo position. Finally, we support our theory by numerical calculations for a rotary propeller towing a spherical cargo.

Let us consider an arbitrary microswimmer (i.e., a propeller) dragging a distant load. In this case, we can neglect the mutual hydrodynamic interaction and calculate the dragging efficiency as

$$\varepsilon_d = \frac{K_l V_d^2}{P_d}, \qquad (1)$$

where K_l is the resistance coefficient of the load [14], V_d is the dragging velocity, and P_d is the rate-of-work expended by the swimmer to drag the load with velocity V_d. We also define the propeller's efficiency in the same fashion,

$$\varepsilon_s = \frac{K_s V_s^2}{P_s},$$

where K_s is the swimmer's resistance coefficient, V_s is the speed of the unloaded propeller (at the point where the load is anchored), and P_s is the power expended in swimming without load.

Note that in a general case of the swimmer propelled by a sequence of geometrically nonreciprocal periodic strokes (e.g., three-link [4] or N-link [5] Purcell's swimmer, surface deformations [2,3], three-sphere swimmer [6], push-me-pully-you [7], etc.), the swimming efficiency is conventionally defined using stroke-averaged quantities [12]. However, since $\max\{K_s V_s^2/P_s\} > K_s\langle V_s\rangle^2/\langle P_s\rangle$ (where $\langle\ \rangle$ stands for the average over a stroke period and the maximum is taken over the stroke period) the maximum of Eq. (1) over a stroke period is an upper bound for the conventional efficiency. In the case of a swimmer propelled without the shape change (e.g., rotating flagella [1], treadmiller [8], or twirling torus [10]), the two definitions coincide. They are also practically equivalent for swimmers performing small-amplitude strokes, with $K_s \approx$ const. Also, note that Eq. (1) is not just the standard swimming efficiency [12] rewritten for "swimmer +load" as a new swimmer, since we aim to compare the rate-of-work expended in dragging the load by the propeller to that spent by an external force.

We will now calculate the dragging efficiency for a swimmer characterized by a resistance coefficient K_s and swimming efficiency ε_s, dragging a load characterized by a resistance coefficient K_l, which we will assume are both not rotating (it is known [3] that a rotating swimmer is less efficient than a nonrotating one). By Lorentz reciprocity [13], if (v_j, σ_{jk}) and (v'_j, σ'_{jk}) are the velocity and stress fields for two

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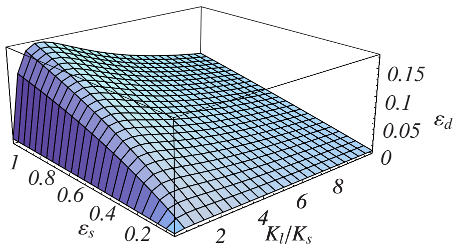

FIG. 1. (Color online) The dragging efficiency, ε_d, as function of the propeller's efficiency ε_s and the size ratio K_l/K_s.

solutions of the Stokes equations $\partial_j\sigma_{ij}=0$ in fluid domain Σ then

$$\int_{\partial\Sigma} v_i'\sigma_{ij}dS_j = \int_{\partial\Sigma} v_i\sigma_{ij}'dS_j. \tag{2}$$

Using Eq. (2) with (v_i,σ_{ij}) being the velocity and the stress fields for a swimmer dragging a load, and (v_i',σ_{ij}') being the velocity and stress fields for the "unloaded" swimmer and the load codragged by the external force with the swimmer's velocity, we readily obtain

$$P_d = P_{s+l} - (V_s - V_d)F_{s+l}. \tag{3}$$

Here P_d is the power expended by the swimmer to drag the load, P_{s+l} is the power dissipated by viscosity in the case of the unloaded swimmer and the load codragged by the external force, V_s is the velocity of the free swimmer, V_d is the dragging velocity, and F_{s+l} is the force required to tow the load with velocity V_s. V_d can be found by equating the sum of the viscous drag forces on the swimmer and the load to zero. Exploiting the linearity of Stokes equation and neglecting hydrodynamic interaction, we obtain

$$V_d = \frac{V_sK_s}{K_s+K_l}. \tag{4}$$

As expected, V_d goes to zero for infinity large load and to the swimmer velocity for a vanishingly small load. Neglecting hydrodynamic interaction, we can use $F_{s+l}=-V_sK_l$ and $P_{s+l}=P_s+V_s^2K_l$, that together with Eq. (4) and dragging power reads

$$P_d = P_s + V_s^2\frac{K_lK_s}{K_l+K_s}. \tag{5}$$

For small loads, $K_l \ll K_s$, Eq. (5) gives the power of the free swimmer plus the power of dragging the load, and for large loads, $K_l \gg K_s$, this gives the power of an anchored swimmer (i.e., a "pump" [9]). Substituting Eq. (3), the swimmer efficiency and Eq. (4) into Eq. (1) gives

$$\varepsilon_d = \frac{r}{(r+1)\left(\dfrac{r+1}{\varepsilon_s}+r\right)}, \tag{6}$$

where $r=K_l/K_s$. The dependence of the dragging efficiency ε_d on ε_s and r is plotted in Fig. 1. Equation (6) shows that unlike the swimming efficiency, which, for some swimmers, can be arbitrarily high [7,8], the dragging efficiency is bounded by $\varepsilon_d \le \frac{1}{r+1} < 1$ even for $\varepsilon_s=\infty$. [The fact that the dragging efficiency must have an optimum can be seen from a simple scaling argument: for $K_l\to 0$, the dragging power reaches linearly to zero, since the dragging speed is constant (equal to the swimmer speed), while the power used by the swimmer is not zero, so $\varepsilon\to 0$. For $K_l\to\infty$, the power used by the swimmer to tow the cargo is, again, not zero (equal to the power of the "pump"), and the dragging speed (4) vanishes similar to $1/K_l$, so the numerator of Eq. (1) goes to zero. Thus, ε_d vanishes at both limits $K_l\to 0$ and at $K_l\to\infty$, and it, therefore, must have a maximum at some finite K_l [15]]. This means that enclosing a cargo *within* the swimmer can be much more efficient than towing a remote one, and that there is an optimal swimmer size for *any* swimming technique (including swimming techniques in which r is varying periodically). As one might expect, ε_d is a growing function of ε_s. However, while for an inefficient propeller (such as a rotating helix) the optimal size is about the same as the load size, the efficient swimmer with $\varepsilon_s \gg 1$ (e.g., push me pull you [7]) will be efficient as the propeller only if it is much larger than the load. Thus, the naive intuition saying that the swimmer's efficiency alone controls the dragging efficiency is not always right: in some cases a less efficient but bigger propeller is advantageous.

Now let us estimate the effect of hydrodynamic interaction between the propeller and the passive cargo separated by distance d. For finite separation distance it is no longer valid to assume that $F_{s+l}=-V_sK_l$. However, the force must still be linear in the dragging velocity and we can write $F_d=\lambda_lK_lV_d$. In the same way, the force on the swimmer must be proportional to the changes in the velocity, so we will denote it by $F_s=-\lambda_s(V_s-V_d)K_s$. Since the forces must still sum up to zero, the dragging velocity is

$$V_d = \frac{V_sK_s}{K_s+\dfrac{\lambda_l}{\lambda_s}K_l}. \tag{7}$$

Comparing the velocity in Eq. (7) to that with no hydrodynamic interaction (4), it can be readily seen that the deviation between the two depends on the ratio λ_l/λ_s: if $\lambda_l/\lambda_s>1$ the velocity will be lower than that in Eq. (4), and if $\lambda_l/\lambda_s<1$ the velocity will be higher than the infinite distance case. Since generally $\lambda_s,\lambda_l<1$ [13], and for asymmetric configurations the resistance coefficient of the larger object will be almost constant, we can conclude that *a large swimmer will drag faster when positioned close to the load, while a small swimmer will drag faster when located far from the load.*

Assuming that the separation distance is large enough, so $d>\max\{R_l,R_s\}$, where R_l and R_s are the hydrodynamic radii of the load and the propeller, respectively, we can now estimate the power needed for the swimmer to drag the cargo: it is known [9] that for any swimmer $P_s=P_p-P_g$, where P_p is the power needed by the pump, i.e., the anchored swimmer, P_s is the power needed by the swimmer when it is swimming freely and P_g is the power needed to drag a "frozen" swimmer with the swimming velocity. If we use this relation by treating the swimmer plus the load as a modified swimmer, we can estimate the power needed to drag the load. In this

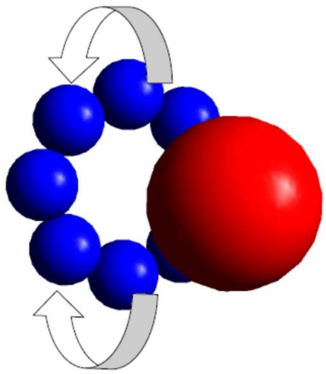

FIG. 2. (Color online) Schematic of the necklacelike propeller towing a spherical load. The arrows show the direction of the rotation of spheres in the propeller; the propeller is pushing the load in front of it.

case, P_g is just the power needed to drag both the load and the (frozen) swimmer with velocity V_d provided by Eq. (7), which is

$$P_g = P_{g(s)}\left(\frac{\lambda_s}{\frac{K_l}{K_s}\frac{\lambda_l}{\lambda_s}+1}\right),$$

where $P_{g(s)}=V_s^2K_s$ is the power needed to drag the immobile swimmer. As an approximation, we will assume that the power expanded by the pump does not depend on the proximity of the load, since $P_p=P_{p(s)}+\mathcal{O}[(R/d)^2]$ ($P_{p(s)}$ is the power expanded by the pump when the load is absent). Together, this gives

$$P_d = P_s + V_s^2K_s\left(1-\frac{\lambda_s}{\frac{K_l}{K_s}\frac{\lambda_l}{\lambda_s}+1}\right), \tag{8}$$

where P_s is the power expended by the swimmer when the load is absent. Obviously, $P_d \geq P_s$ and the equality holds only when $K_l=0$.

Substitution of Eqs. (8) and (7) in Eq. (1) yields

$$\varepsilon_d = \frac{r}{\left[\left(\frac{\lambda_l r}{\lambda_s}+1\right)\left(1+\frac{1}{\varepsilon_s}\right)-\lambda_s\right]\left(\frac{\lambda_l r}{\lambda_s}+1\right)}, \tag{9}$$

where $r=K_l/K_s$. For $\lambda_s=\lambda_l=1$ Eq. (9) reduces to Eq. (6), as anticipated. Comparing the efficiency in Eq. (9) to that in Eq. (6), one can conclude that in cases where $\lambda_l/\lambda_s>1$ (i.e., the swimmer is smaller than the load), the efficiency is lower when the hydrodynamic coupling is not negligible, and it would be better separated from the load. If the swimmer is much bigger than the load, which implies $\lambda_s \simeq 1$ and $\lambda_l/\lambda_s<1$, the efficiency is higher than in the case with no coupling. Thus a propeller bigger than the load should be positioned closer to the load, opposite to Purcell's assumption [1]. Equation (9) also tells us that the efficiency is bounded by λ_s/λ_l, which can theoretically be greater than 1 for a large propeller towing a small load. However, we could not find such an example.

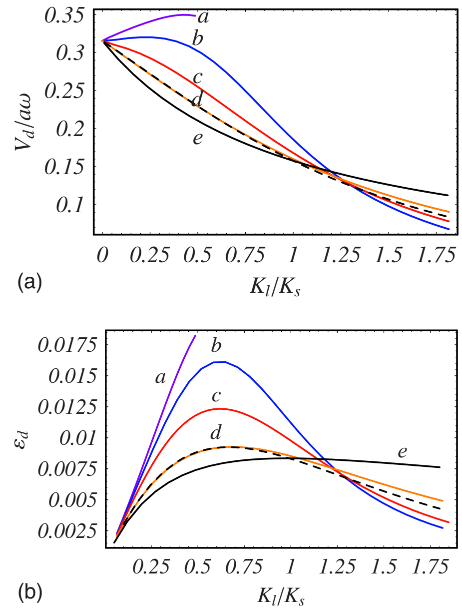

FIG. 3. (Color online) Numerical results for the "necklace-shaped" propeller made of eight corotating spheres of radius a ($V_s/a\omega \simeq 0.316, \varepsilon_s=0.0339, R_s=3.083a$), towing a spherical load of variable size located at $d_*=0$ (magenta, a), $d_*=a$ (blue, b), $d_*=4a$ (red, c) and $d_*=10a$ (yellow, d); the solid line e corresponds to the infinite-separation result (4); the dashed lines are the far-field approximations for $d_*=10a$. (a) The scaled dragging speed, $V_d/a\omega$; (b) The dragging efficiency, ε_d.

We can now estimate λ_s and λ_l as functions of d, using the Oseen tensor [13]. As the first order approximation, we will assume both the swimmer and the load can be modeled as spheres with hydrodynamic radii $R_s=K_s/6\pi\mu$ and $R_l=K_l/6\pi\mu$, respectively [16]. In this case, it can be readily shown [13] that for $d \gg \max\{R_l,R_s\}$,

$$\lambda_s \approx \frac{2(2d^2-3dR_l)}{4d^2-9R_lR_s}, \quad \lambda_l \approx \frac{2(2d^2-3dR_s)}{4d^2-9R_lR_s}.$$

Substituting these expressions into Eqs. (7) and (9) gives the leading approximation for the dragging speed and efficiency, respectively, as a function of dimensionless proximity $\delta = d/R_s$ and the size ratio r. Expanding the resulting expression for the dragging efficiency for small $\frac{1}{\delta}$ gives

$$\varepsilon_d \approx \varepsilon_{d(\infty)} + \frac{3\varepsilon_{d(\infty)}^2}{\varepsilon_s\delta}\left[1-(1+\varepsilon_s)r^2\right] + \cdots,$$

where $\varepsilon_{d(\infty)}$ corresponds to the no-hydrodynamic-interaction approximation for the dragging efficiency (6). The $1/\delta$ term in the above expansion shows that for $r>\frac{1}{\sqrt{1+\varepsilon_s}}$ the dragging is retarded in comparison to the infinite separation result (6), i.e., $\varepsilon_d<\varepsilon_{d(\infty)}$, while for $r<\frac{1}{\sqrt{1+\varepsilon_s}}$, the dragging is enhanced due to the hydrodynamic coupling, as $\varepsilon_d>\varepsilon_{d(\infty)}$. Interestingly, $r=\frac{1}{\sqrt{1+\varepsilon_s}}$ corresponds to the maximum of $\varepsilon_{d(\infty)}$. However, it is not the optimum of ε_d, which shifts to higher values at smaller r's.

We shall now test the proposed theory for the load dragged by a rotary propeller. Imagine the necklacelike ring (see Fig. 2) of $N_p=8$ nearly touching rigid spheres (separated by the distance of $0.05a$) of radius a. The necklace lies in the xy plane and in a cylindrical polar coordinate system (z,r,φ), each sphere rotates at the constant angular velocity $\boldsymbol{\Omega}=\omega \boldsymbol{e}_\varphi$, which, in the absence of external forces, causes the necklace to swim along the normal to the plane of the necklace in the positive z direction [10]. Performance of this swimmer as a cargo propeller is tested for a spherical particle positioned at arbitrary distance along the z axis [17]. The distance that separates the plane of the propeller ($z=0$) and the load's surface is denoted by d_*. We use the method of Ref. [10] and construct the rigorous solution of the Stokes equations as superposition of Lamb's spherical harmonic expansions [13]. The no-slip conditions at the surface of all spheres are enforced via the direct transformation between solid spherical harmonics centered at origins of different spheres. The accuracy of the calculations is controlled by the number of spherical harmonics, L, retained in the series. The truncation level of $L\leq 7$ was found to be sufficient for all configuration to achieve an accuracy of less than 1%. The dragging efficiency (1) for this particular swimmer reads $\varepsilon_d=K_l V_d^2/N_p T\omega$, where T is a hydrodynamic torque exerted on each sphere of the propeller towing the load. The values of K_l, T, and V_d are determined numerically and the resulting dragging speed $V_d/a\omega$, and efficiency, ε_d, are plotted vs the size ratio r in Figs. 3(a) and 3(b), respectively. The agreement with the far-field asymptotic results (7) and (9) (via $\delta=d_*/R_s+r$) is excellent for small loads ($r<1$) even at moderate proximity of $d_*=10a$ (i.e., $\delta\simeq 3.24$). It can be readily seen that there is an optimal swimmer-load size ratio in all cases. Interestingly, Fig. 3(a) shows that, while for moderate separation the dragging velocity decays with the increase in the load size, at close proximity it may actually become higher than the velocity of the unloaded swimmer. This is a direct consequence of Eq. (7), which does not assume large separation: the fluid velocity in the center of the "necklace" is larger than the swimming speed. This means that in order to pull a load positioned at $d_*=0$ with the swimmer's speed, the applied force must act in the direction opposite to that of the velocity, so that $\lambda_l<0$ and $V_d>V_s$. The numerical results confirm the qualitative dependencies arising from the far-field theory: there is a critical size-ratio r_{cr} (weakly dependent on δ) such that for $r<r_{cr}$ the dragging efficiency is higher than the corresponding $\varepsilon_{d(\infty)}$ and for $r>r_{cr}$ the efficiency is lower than $\varepsilon_{d(\infty)}$; at moderate separations $r_{cr}\to 1/\sqrt{1+\varepsilon_s}$ as expected from the far-field analysis. The discrepancy between the asymptotic and the numerical results is only observed at $r>1$, where the assumption $\delta/r\gg 1$ is no longer valid.

To conclude, we investigated the dragging efficiency of an arbitrary swimmer towing a cargo at low Reynolds numbers. It was demonstrated that there is an optimal hydrodynamic size ratio of the propeller and the cargo. The dragging efficiency and the size ratio at the optimum depend upon the propeller-load mutual proximity.

This work was supported by the Technion V.P.R. Fund. We thank J. E. Avron for fruitful discussions.

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[14] We treat K_l as a scalar, since the resistance matrix is symmetric [13], and it is optimal to drag the load when the velocity is aligned in the direction of the minimal eigenvalue.

[15] We thank the anonymous referee for this comment.

[16] If either load or propeller deviates considerably from the spherical shape, it can be taken care of in the far-field resistance tensor. Here, for simplicity, we only refer to the far-field interaction between two spheres.

[17] For computational simplicity, the hydrodynamic disturbance caused by the links required to connect the propeller and the load is neglected.